TRUNCATED PUSHFORWARDS AND REFINED UNRAMIFIED COHOMOLOGY

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Abstract. For a large class of cohomology theories, we prove that refined unramified cohomology is canonically isomorphic to the hypercohomology of a natural truncated complex of Zariski sheaves. This generalizes a classical result of Bloch and Ogus and solves a conjecture of Kok and Zhou.

1. Introduction

Let X be a smooth variety over a field k. We consider the natural map $\pi : X_{\text{ét}} \to X_{\text{Zar}}$ between the étale site and the Zariski site of X. Let m be an integer invertible in k. Then the total pushforward $\mathrm{R}\,\pi_*\mu_m^{\otimes n} \in D^b(X_{\text{Zar}})$ is known to capture the motivic cohomology of X with values in $\mathbb{Z}/m$. Indeed, as a consequence of the Bloch–Kato conjecture, proven by Voevodsky [Voe11], we have

$$H^i_M(X, \mathbb{Z}/m(n)) \simeq H^i(X, \tau_{\leq n}\,\mathrm{R}\,\pi_*\mu_m^{\otimes n}), \tag{1.1}$$

see [GL01, Corollary 1.2]. Since $H^i_M(X, \mathbb{Z}/m(n)) \simeq \mathrm{CH}^n(X, 2n-i, \mathbb{Z}/m)$ agrees with Bloch's higher Chow groups (with finite coefficients), it also follows that the Zariski cohomology of the truncated complex $\tau_{\leq n}\,\mathrm{R}\,\pi_*\mu_m^{\otimes n}$ has an explicit geometric meaning in terms of algebraic cycles on $X \times \Delta^q$. Similar results hold for $m = p^r$, when k is a perfect field of characteristic p, see [GL00]. If k contains a primitive m-th root of unity, then $\mu_m^{\otimes n}$ can be replaced by $\mathbb{Z}/m$ in this discussion.

It is natural to wonder whether the hypercohomology of "the other" truncation $\tau_{\geq s}\,\mathrm{R}\,\pi_*\mu_m^{\otimes n}$ admits a natural geometric description as well. In a recent paper, Kok and Zhou [KZ23] found the following conjectural answer to this question.

Conjecture 1.1 ([KZ23, Conjecture 1.10])**.** *Let X be a smooth variety over a field k and let $A(n)$ be a (twisted) locally constant torsion étale sheaf in which the exponential characteristic of k is invertible. Then there are canonical isomorphisms*

$$H^i(X, \tau_{\geq s}\,\mathrm{R}\,\pi_*A(n)) \simeq H^i_{i-s,nr}(X, A(n)),$$

where the right hand side denotes the refined unramified cohomology groups from [Sch23] and where $\pi : X_{ét} \to X_{Zar}$ denotes the canonical map. Moreover, if $k = \mathbb{C}$ and $X_{ét}$ is replaced by the analytic site X_{an}, then $A(n)$ can be replaced by any abelian group.

We recall that the refined unramified cohomology groups are defined by

$$H^i_{j,nr}(X, A(n)) = \mathrm{im}(H^i(F_{j+1}X, A(n)) \to H^i(F_jX, A(n))),$$

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where $F_jX = \{x \in X \mid \operatorname{codim}(x) \leq j\}$ and $H^i(F_jX, A(n)) = \operatorname{colim}_{U\supset F_jX} H^i(U, A(n))$. Equivalently, $H^i_{j,nr}(X, A(n))$ is the subgroup of $H^i(F_jX, A(n))$, given by those classes that have trivial residues at all codimension $j+1$ points, see [Sch23, Lemma 5.8]. In particular, for $j = 0$, this definition naturally recovers classical unramified cohomology, see [CT95, Theorem 4.1.1(a)] and [CTO89]. Other special cases are ordinary cohomology (for $j \geq \dim X$) or Kato homology, see [Sch23, §1.3].

The case $i = s$ of Conjecture 1.1 implies by the hypercohomology spectral sequence the isomorphism

$$H^i_{0,nr}(X, A(n)) \simeq H^0(X_{\mathrm{Zar}}, \mathrm{R}^i\, \pi_* A(n)).$$

This is a celebrated result of Bloch and Ogus [BO74], which identifies the unramified cohomology of a smooth variety by the global sections of a certain Zariski sheaf. Conjecture 1.1 may be seen as a generalization of this result. This has, as we shall discuss below, various interesting consequences.

The purpose of this paper is to prove (a generalization of) Conjecture 1.1. To state our result, consider the following examples of a Grothendieck topology ν on a k-scheme X and a complex $K^\bullet \in D(X_\nu, \mathbb{Z})$ of sheaves of abelian groups on X_ν:

(1) k any field, $\nu =$ ét the small étale site of X, and $K^\bullet$ the pullback of a bounded below complex of (arbitrary) étale sheaves on $\operatorname{Spec} k$.
(2) $k = \mathbb{C}$, $\nu =$ an the analytic site on $X(\mathbb{C})$ and $K^\bullet$ any bounded below complex of constant sheaves of abelian groups.
(3) k any field of characteristic zero, $\nu =$ Zar the Zariski site on X and $K^\bullet = \Omega^\bullet_{X/k}$ the algebraic de Rham complex of X over k.
(4) k any perfect field of characteristic $p > 0$, $\nu =$ ét the small étale site and $K^\bullet$ (some shift of) the logarithmic de Rham Witt sheaf $W_r\Omega^n_{X,\log}$, see [Il79].
(5) k any perfect field of characteristic $p > 0$, $\nu =$ Zar the small Zariski site and $K^\bullet$ the de Rham Witt complex $W\Omega^\bullet_X := \varprojlim W_n\Omega^\bullet_X$ of X, see [Il79].
(6) k any field, $\nu =$ proét the small pro-étale site of Bhatt–Scholze and $K^\bullet$ the pullback of a constructible complex in $D_{\mathrm{cons}}((\operatorname{Spec} k)_{\text{proét}}, \widehat{\mathbb{Z}}_\ell)$, see [BS15].
(7) k a perfect field, $\nu =$ Zar the Zariski site and
$$K^\bullet := A_X(n)_{\mathrm{Zar}} := z^n(-_{\mathrm{Zar}}, \bullet)[-2n] \otimes^{\mathbb{L}} A$$
Bloch's cycle complex with values in an abelian group A, see [Blo86].
(8) k any field, $\nu =$ ét the étale site, A an abelian group in which the characteristic exponent p of k is invertible and
$$K^\bullet := A_X(n)_{\text{ét}} := z^n(-_{\text{ét}}, \bullet)[-2n] \otimes^{\mathbb{L}} A$$
the étale sheafification of Bloch's cycle complex with values in A.

Theorem 1.2. *Let X be a smooth equi-dimensional algebraic k-scheme. Let ν be a Grothendieck topology on X that contains all Zariski open covers and let $K^\bullet \in D(X_\nu, \mathbb{Z})$ be a complex of abelian sheaves as in examples (1)–(8) above. Let $\pi : X_\nu \to X_{Zar}$ be the natural morphism of sites. Then for all integers i, s, there is a canonical isomorphism:*

$$H^i(X_{Zar}, \tau_{\geq s}\, \mathrm{R}\, \pi_* K^\bullet) \simeq H^i_{i-s,nr}(X_\nu, K^\bullet),$$

where $H^i_{j,nr}(X_\nu, K^\bullet) := \operatorname{im}(H^i(F_{j+1}X, K^\bullet) \to H^i(F_jX, K^\bullet))$ and $H^i(F_jX, K^\bullet) = \operatorname{colim}_{U\supset F_jX} H^i(U, K^\bullet)$.

Theorem 1.2 applied to (special cases of) examples (1) and (2) proves Conjecture 1.1.

For a large class of site theoretic cohomology theories, the theorem yields an explicit geometric interpretation of the Zariski cohomology $H^i(X, \tau_{\geq s} \mathrm{R}\, \pi_* K^\bullet)$ of the truncated pushforward $\tau_{\geq s} \mathrm{R}\, \pi_* K^\bullet$ via (unramified) cohomology classes on some open subsets of X, thereby complementing the aforementioned description of $H^i(X, \tau_{\leq n} \mathrm{R}\, \pi_* \mu_m^{\otimes n})$ in terms of higher Chow groups with finite coefficients. In particular, we get the following, where

$$H^i_M(X, \mathbb{Z}/m(n)) := H^i(X_{\mathrm{Zar}}, (\mathbb{Z}/m)_X(n)_{\mathrm{Zar}}) \quad \text{and} \quad H^i_L(X, \mathbb{Z}/m(n)) := H^i(X_{\text{ét}}, (\mathbb{Z}/m)_X(n)_{\text{ét}})$$

denote motivic and étale motivic (or Lichtenbaum) cohomology, respectively.

Corollary 1.3. *Let X be a smooth equi-dimensional algebraic scheme over a perfect field k and let m be an arbitrary positive integer. Then there is a natural long exact sequence*

$$\cdots \to H^i_M(X, \mathbb{Z}/m(n)) \stackrel{\mathrm{cl}}{\to} H^i_L(X, \mathbb{Z}/m(n)) \to H^i_{i-n-1,nr}(X_{\acute{e}t}, (\mathbb{Z}/m)_X(n)_{\acute{e}t}) \to H^{i+1}_M(X, \mathbb{Z}/m(n)) \stackrel{\mathrm{cl}}{\to} \cdots .$$

By work of Geisser–Levine [GL00, GL01], we have canonical quasi-isomorphisms

$$(\mathbb{Z}/m)_X(n)_{\text{ét}} \simeq \begin{cases} \mu_m^{\otimes n} & \text{if } m \text{ is coprime to } \mathrm{char}(k); \\ W_r\Omega^n_{X,\log}[-n] & \text{if } m = p^r \text{ and } p = \mathrm{char}(k) > 0. \end{cases}$$

Hence the groups $H^i_L(X, \mathbb{Z}/m(n))$ identify to étale cohomology and logarithmic de Rham Witt cohomology, respectively, and the change of topology map cl may be identified with a natural cycle class map for motivic cohomology (resp. higher Chow groups) with finite coefficients. Corollary 1.3 shows that the kernel and cokernel of these cycle class maps are controlled by refined unramified cohomology. The result was proven for k algebraically closed and m coprime to the characteristic by Kok and Zhou [KZ23] (which partly motivated their Conjecture 1.1).

Away from the characteristic, refined unramified cohomology was previously known to be closely related to algebraic cycles [Sch23], generalizing various previous results for ordinary unramified cohomology and cycles of low (co-)dimensions from [CTV12, Kah12, Voi12, Ma17]. Comparison results between algebraic cycles and refined unramified cohomology, together with the fact that the latter can be represented by explicit cohomology classes on certain open subsets of X, have already seen various applications, for instance to prove nontriviality of torsion classes in Griffiths groups [Sch24, Ale23], to prove non-algebraicity of certain Tate and Hodge classes [Kok23], and to study zero-cycles over non-closed fields [AS23, Ale24]. However, the previous techniques did not allow to tackle the case of p-torsion coefficients in characteristic p, partly because the strong form of purity that is used in all of the above works fails for logarithmic de Rham Witt cohomology, see e.g. [G85, p. 45, Remarque].

In the case of logarithmic de Rham Witt cohomology, Theorem 1.2 yields the following explicit calculations for the refined unramified cohomology groups.

Corollary 1.4. *Let k be a perfect field of positive characteristic $p>0$. Let X be a smooth and equi-dimensional algebraic k-scheme. Then for all integers i,j, we have canonical isomorphisms*

$$H^i_{j,nr}(X_{\acute{e}t},(\mathbb{Z}/p^r)_X(n))\simeq\begin{cases}H^i_L(X,\mathbb{Z}/p^r(n)), & \text{if } j\geq i-n\\ H^j(X_{Zar},\mathrm{R}^{i-j}\,\pi_*(\mathbb{Z}/p^r)_X(n)), & \text{if } j=i-n-1\\ 0, & \text{if } j\leq i-n-2,\end{cases} \tag{1.2}$$

where the complex $(\mathbb{Z}/p^r)_X(n)$ denotes the $-n$-shifted logarithmic de Rham Witt sheaf $W_r\Omega^n_{X,\log}$ from [Il79] *and where $\pi: X_{\acute{e}t}\to X_{Zar}$ is the natural map of sites induced by the identity.*

Another direct application of Theorem 1.2 is as follows.

Corollary 1.5. *Let $f:X\to Y$ be a morphism between equi-dimensional smooth algebraic k-schemes and let k, ν and $K^\bullet_Y\in D(Y_\nu,\mathbb{Z})$ be as in one of the examples* (1)–(8) *above. Let $K^\bullet_X\in D(X_\nu,\mathbb{Z})$ be the analogous complex on X_ν. Then for all i,j there is a functorial pullback map*

$$f^*: H^i_{j,nr}(Y,K^\bullet_Y)\longrightarrow H^i_{j,nr}(X,K^\bullet_X),$$

which coincides with the canonical pullback $H^i(X,K^\bullet_Y)\to H^i(X,K^\bullet_X)$ for $j\geq\max\{\dim X,\dim Y\}$.

The existence of pullbacks for refined unramified cohomology is a priori not clear. Indeed, a class in $H^i_{j,nr}(Y,K^\bullet_Y)$ is represented by an unramified cohomology class on an open subset V of Y whose complement has codimension $j+1$ in Y. This class can be pulled back to a class on $U:=f^{-1}(V)$, but unless f is flat, it is not clear that the complement of U in X has codimension $j+1$. Hence there is no direct way to define pullbacks. This issue has been solved in [Sch22] for various cohomology theories (including ℓ-adic (pro-)étale cohomology) of quasi-projective varieties in characteristic zero via a moving lemma for cohomology with support and in [KZ24] for ℓ-adic étale cohomology of any smooth variety via deformation to the normal cone. The above result is more general and covers in particular the case of p-torsion coefficients in characteristic p, which is new. In fact, the existence of functorial pullbacks for refined unramified cohomology in the case of non homotopy invariant examples e.g. the example (5) which computes crystalline cohomology, does not follow from the methods considered in [KZ24], see [KZ24, Definition 2.4].

In most of the examples (1)–(8), it is easy to see the existence of pushforwards and exterior products for refined unramified cohomology (via its very definition, not via Theorem 1.2). Combining this with the pullbacks from Corollary 1.5 one can reprove the results in [Sch22, KZ24]. Our argument is slightly more general and allows us to treat for instance refined unramified cohomology $H^i_{j,nr}(X_{\mathrm{Zar}},W\Omega^\bullet_X)$ associated to the complex in (5), which computes integral crystalline cohomology (see [Il79, p. 606, Théorème II.1.4]) and is not covered in [Sch22, KZ24].

Corollary 1.6. *Let k be a perfect field of characteristic $p>0$. Let X and Y be smooth, proper and equi-dimensional algebraic schemes over k. If we set $d_X:=\dim X$, then for all $c,i,j\geq 0$, there is a bi-additive pairing*

$$\mathrm{CH}^c(X\times Y)\times H^i_{j,nr}(X,W\Omega^\bullet_X)\longrightarrow H^{i+2c-2d_X}_{j+c-d_X,nr}(Y,W\Omega^\bullet_Y),\quad ([\Gamma],\alpha)\longmapsto[\Gamma]_*(\alpha), \tag{1.3}$$

which is functorial with respect to the composition of correspondences.

Theorem 1.2 will be deduced from the following more general result, which applies essentially to any site theoretic cohomology theory that satisfies a version of the Gersten conjecture.

Theorem 1.7. *Let X be an equi-dimensional algebraic k-scheme. Let $\pi : X_\nu \to X_{Zar}$ be a morphism of sites associated to some Grothendieck topology ν on (Sch/k) that contains all Zariski open coverings. Suppose that there is a complex $K^\bullet \in D(X_\nu, \mathbb{Z})$ such that for all i, there is a resolution $\epsilon^i : \mathrm{R}^i \pi_* K^\bullet \to \mathcal{E}_X^{i,\bullet}$ in $D^+(X_{Zar}, \mathbb{Z})$ by a complex $\mathcal{E}_X^{i,\bullet}$ concentrated in non-negative degrees of the form:*

(1.4)
$$\mathcal{E}_X^{i,\bullet} : \ 0 \longrightarrow \bigoplus_{x\in X^{(0)}} \iota_{x*}\underline{A}_x^i \longrightarrow \bigoplus_{x\in X^{(1)}} \iota_{x*}\underline{A}_x^{i+1} \longrightarrow \bigoplus_{x\in X^{(2)}} \iota_{x*}\underline{A}_x^{i+2} \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(r)}} \iota_{x*}\underline{A}_x^{i+r} \longrightarrow \cdots ,$$

where $\iota_{x*}\underline{A}_x^{i+j}$ placed in degree j is a constant sheaf supported on $\overline{\{x\}} \subset X$ which corresponds to some abelian group A_x^{i+j}.

Then for all integers i, s, there is a canonical isomorphism:

$$H^i(X_{Zar}, \tau_{\geq s} \mathrm{R}\,\pi_* K^\bullet) \simeq H^i_{i-s,nr}(X_\nu, K^\bullet), \tag{1.5}$$

where $H^i_{j,nr}(X_\nu, K^\bullet) := \mathrm{im}(H^i(F_{j+1}X, K^\bullet) \to H^i(F_jX, K^\bullet))$ *and* $H^i(F_jX, K^\bullet) = \mathrm{colim}_{U\supset F_jX} H^i(U, K^\bullet)$.

Note that the resolution in (1.4) is automatically acyclic, because the sheaves $\iota_{x*}\underline{A}_x^{i+j}$ are flasque on the Zariski site of X.

In practice, the resolution (1.4) that we consider is given by instances where Gersten's conjecture holds, see e.g. [Qui73, BO74, GS88, CTHK97, Sch22]. This requires X to be smooth, even though the formal set-up in the above theorem does not need this requirement.

2. Notation, conventions, and preliminaries.

The exponential characteristic of a field k of characteristic $p \geq 0$ is 1 if $p = 0$ and it is p otherwise.

An algebraic scheme is a separated scheme of finite type over a field. A variety is an integral algebraic scheme.

If X is a scheme and ν denotes a Grothendieck topology on X, then $D(X_\nu, \mathbb{Z})$ denotes the derived category of the abelian category of sheaves of abelian groups on X_ν. We further denote by $D^+(X_\nu, \mathbb{Z})$ the full triangulated subcategory that consists of objects that have trivial cohomology sheaves in sufficiently negative degrees. For a complex $K^\bullet \in D(X_\nu, \mathbb{Z})$, we denote by $\mathcal{H}^n(K^\bullet)$ its n-th cohomology sheaf.

Recall that for any complex $K^\bullet \in D(X_\nu, \mathbb{Z})$ there is a K-injective replacement $K^\bullet \to I^\bullet$, see [Stacks24, Tag 079P]. Using this, any left exact functor can be derived, see [Stacks24, Tag 079V]. For a complex $K^\bullet \in D(X_\nu, \mathbb{Z})$ and a closed subset $Z \subset X$, we use the notations

$$H^i(X_\nu, K^\bullet) := R^i\Gamma(X_\nu, K^\bullet) \quad \text{and} \quad H^i_Z(X_\nu, K^\bullet) := R^i\Gamma_Z(X_\nu, K^\bullet),$$

where Γ and Γ_Z denote the global section functor, and the global section functor with support, respectively. If $K^\bullet = \mathcal{F}[0]$ is a sheaf placed in degree zero, then we write $H^i(X_\nu, \mathcal{F}) := H^i(X_\nu, \mathcal{F}[0])$. If $j : U \hookrightarrow X$ is an open immersion, then we write

$$H^i(U_\nu, K^\bullet) := H^i(U_\nu, j^* K^\bullet).$$

We write $F_jX := \{x \in X \mid \operatorname{codim}(x) \leq j\}$, where $\operatorname{codim}(x) := \dim X - \dim \overline{\{x\}}$. This may be seen as a pro-scheme which consists of all open subsets of X that contain all codimension j points of X. For any Grothendieck topology ν on (Sch/k) and any complex $K^\bullet \in D(X_\nu, \mathbb{Z})$, we define the group

$$H^i(F_jX, K^\bullet) := \varinjlim H^i(U_\nu, K^\bullet), \tag{2.1}$$

where the direct limit is taken over all Zariski open subsets $U \subset X$ with $F_jX \subset U$. If $U \subset V \subset X$ are Zariski open subsets, then there are canonical restriction maps $H^i(V_\nu, K^\bullet) \to H^i(U_\nu, K^\bullet)$. These maps induce natural restriction maps

$$H^i(F_{j+1}X, K^\bullet) \longrightarrow H^i(F_jX, K^\bullet).$$

In analogy to [Sch23, Definition 5.1], we define the image of this map as the j-th refined unramified cohomology of the complex $K^\bullet \in D(X_\nu, \mathbb{Z})$ by

$$H^i_{j,nr}(X, K^\bullet) := \operatorname{im}\left(H^i(F_{j+1}X, K^\bullet) \to H^i(F_jX, K^\bullet)\right). \tag{2.2}$$

3. Proof of Theorem 1.7

The following is Theorem 1.7 with a slightly different but equivalent formulation.

Theorem 3.1. *Let X be an equi-dimensional algebraic k-scheme. Let $K^\bullet \in D(X_{Zar}, \mathbb{Z})$ be a complex, such that for all i, there is a resolution $\epsilon^i : \mathcal{H}^i(K^\bullet) \to \mathcal{E}_X^{i,\bullet}$ in $D^+(X_{Zar}, \mathbb{Z})$ of the i-th Zariski cohomology sheaf of $K^\bullet$ by a complex $\mathcal{E}_X^{i,\bullet}$ concentrated in non-negative degrees of the form:*

$$\mathcal{E}_X^{i,\bullet}:\ 0 \longrightarrow \bigoplus_{x\in X^{(0)}} \iota_{x*}\underline{A}_x^i \longrightarrow \bigoplus_{x\in X^{(1)}} \iota_{x*}\underline{A}_x^{i+1} \longrightarrow \bigoplus_{x\in X^{(2)}} \iota_{x*}\underline{A}_x^{i+2} \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(r)}} \iota_{x*}\underline{A}_x^{i+r} \longrightarrow \cdots, \tag{3.1}$$

where $\iota_{x*}\underline{A}_x^{i+j}$ placed in degree j is a constant sheaf supported on $\overline{\{x\}} \subset X$ which corresponds to some abelian group A_x^{i+j}.

Then for all integers i, s, there is a canonical isomorphism:

$$H^i(X_{Zar}, \tau_{\geq s}K^\bullet) \simeq H^i_{i-s,nr}(X_{Zar}, K^\bullet), \tag{3.2}$$

where the group $H^i_{j,nr}(X_{Zar}, K^\bullet)$ is the j-th refined unramified cohomology defined in (2.2).

Proof. We prove a series of preparatory lemmas that combined all together yield Theorem 3.1. We begin with the following:

Lemma 3.2. *The natural restriction map*

$$H^i(F_{j+1}X, \tau_{\geq s}K^\bullet) \longrightarrow H^i(F_jX, \tau_{\geq s}K^\bullet)$$

is an isomorphism if $j > i - s$ and injective for $j = i - s$.

Proof. For any closed subscheme $\iota : Z \hookrightarrow X$ of pure codimension $c > 0$ with complement $j_U : U \hookrightarrow X$, we consider the complex $\iota^! \mathcal{E}_X^{q,\bullet} \in D^+(Z_{\mathrm{Zar}}, \mathbb{Z})$ defined by

$$\iota_*\iota^!\mathcal{E}_X^{q,\bullet} := \ker(\alpha_{j_U} : \mathcal{E}_X^{q,\bullet} \longrightarrow j_{U*}j_U^*\mathcal{E}_X^{q,\bullet}), \tag{3.3}$$

where the above kernel is taken in the abelian category of cochain complexes of abelian sheaves on X_{Zar} and the map α_{j_U} is the unit of the relevant adjunction.

By evaluating (3.3) at open subsets of X, the set-theoretic equality $X^{(p)} \setminus U^{(p)} = Z^{(p-c)}$ yields that

$$\iota^! \mathcal{E}_X^{q,p} = \bigoplus_{x \in Z^{(p-c)}} \iota_{x*} \underline{A}_x^{q+p}$$

is a complex of flasque sheaves on the Zariski site. Moreover, we find that $\iota^! \mathcal{E}_X^{q,p} = 0$ in all degrees $p < c$. Hence $\iota^! \mathcal{E}_X^{q,\bullet} = \mathcal{E}_Z^{q+c,\bullet}[-c]$, where $\mathcal{E}_Z^{q+c,\bullet} \in D^+(Z_{\text{Zar}}, \mathbb{Z})$ with

$$\mathcal{E}_Z^{q+c,p} = \bigoplus_{x \in Z^{(p)}} \iota_{x*} \underline{A}_x^{q+c+p}$$

is a complex of the form (3.1) on Z (by no means we claim here that the latter complex is exact in degrees > 0). It follows directly from the definition of the derived functor for the global section functor with support Γ_Z that for any non-empty open subset $V \subset X$, we get natural isomorphisms

$$\mathrm{R}\,\Gamma_{Z \cap V}(V_{\text{Zar}}, \mathcal{H}^q(K^\bullet)) \simeq \Gamma(Z \cap V, \iota^! \mathcal{E}_X^{q,\bullet}) = \Gamma(Z \cap V, \mathcal{E}_Z^{q+c,\bullet})[-c], \tag{3.4}$$

that are contravariantly functorial with respect to V and covariantly functorial with respect to equi-dimensional closed subsets $Z \subset Z' \subset X$ of the same codimension c in X. Note that above we explicitly use that $\mathcal{H}^q(K^\bullet) \to \mathcal{E}_X^{q,\bullet}$ is an acyclic (flasque) resolution for the global section functor with support. In particular, taking cohomology of (3.4) gives:

$$H^p_{Z \cap V}(V_{\text{Zar}}, \mathcal{H}^q(K^\bullet)) \simeq H^{p-c}(\Gamma(Z \cap V, \mathcal{E}_Z^{q+c,\bullet})). \tag{3.5}$$

Now pick a pair (Z, W) of equi-dimensional closed subsets $W \subset Z \subset X$ such that $\dim Z = \dim X - (j+1)$ and $\dim W = \dim X - (j+2)$. We assign to (Z, W) a natural long exact localisation sequence:

$$\cdots \longrightarrow H^i_{Z \setminus W}(X \setminus W, \tau_{\geq s} K^\bullet) \longrightarrow H^i(X \setminus W, \tau_{\geq s} K^\bullet) \longrightarrow H^i(X \setminus Z, \tau_{\geq s} K^\bullet) \longrightarrow \cdots, \tag{3.6}$$

which we denote by $\textbf{L-SEQ}_{(Z,W)}$.

Let $\mathcal{I}$ denote the index set, whose elements are pairs (Z, W) of equi-dimensional closed subsets $W \subset Z \subset X$ as above, i.e. $\dim Z = \dim X - (j+1)$ and $\dim W = \dim X - (j+2)$. We turn $\mathcal{I}$ into a directed set by declaring $(Z', W') \leq (Z, W)$ if and only if $Z \subset Z'$ and $W \subset W'$. It is readily seen that for $(Z', W') \leq (Z, W)$, we get a canonical map $\textbf{L-SEQ}_{(Z,W)} \to \textbf{L-SEQ}_{(Z',W')}$. Indeed, this is achieved by first restricting the sequence $\textbf{L-SEQ}_{(Z,W)}$ to the open subset $X \setminus W'$ and then composing with

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^i_{Z \setminus W'}(X \setminus W') & \longrightarrow & H^i(X \setminus W') & \longrightarrow & H^i(X \setminus (W' \cup Z)) & \longrightarrow & \cdots \\ & & \downarrow & & \| & & \downarrow & & \\ \cdots & \longrightarrow & H^i_{Z' \setminus W'}(X \setminus W') & \longrightarrow & H^i(X \setminus W') & \longrightarrow & H^i(X \setminus Z') & \longrightarrow & \cdots. \end{array}$$

In particular, this gives a direct system over $\mathcal{I}$. Taking the direct limit over this index set, (3.6) yields

$$\cdots \longrightarrow \bigoplus_{x \in X^{(j+1)}} H^i_x(X, \tau_{\geq s} K^\bullet) \longrightarrow H^i(F_{j+1}X, \tau_{\geq s} K^\bullet) \longrightarrow H^i(F_j X, \tau_{\geq s} K^\bullet) \longrightarrow \cdots, \tag{3.7}$$

where we put $H^i_x(X, \tau_{\geq s} K^\bullet) := \varinjlim H^i_{V \cap \overline{\{x\}}}(V, \tau_{\geq s} K^\bullet)$ and the direct limit here runs through opens $V \subset X$ with $x \in V$.

The claim thus follows from (3.7) once we show that the groups $H^i_x(X,\tau_{\geq s}K^\bullet)$ vanish for all $i\leq j+s$ and all points $x\in X^{(j+1)}$. By exactness of the direct limit functor, we obtain a hypercohomology spectral sequence

$$E_2^{p,q}:=H^p_x(X,\mathcal{H}^q(\tau_{\geq s}K^\bullet))\implies H^{p+q}_x(X,\tau_{\geq s}K^\bullet).$$

We clearly have that $E_2^{p,q}=0$ for $q<s$. Thus it is enough to prove that

$$H^p_x(X,\mathcal{H}^q(K^\bullet))=0 \text{ for } q\geq s,\ p+q=i \text{ and } j\geq i-s. \tag{3.8}$$

Indeed, let $c:=j+1$ and fix the notation $Z_x:=\overline{\{x\}}\subset X$ with $x\in X^{(c)}$. Note that by (3.5), we find a natural isomorphism

$$H^p_x(X,\mathcal{H}^q(K^\bullet))\simeq H^{p-c}(\Gamma(F_0Z_x,\mathcal{E}^{q+c,\bullet}_{Z_x})), \tag{3.9}$$

where we set $H^p(\Gamma(F_0Z_x,\mathcal{E}^{q+c,\bullet}_{Z_x})):=\varinjlim H^p(\Gamma(V,\mathcal{E}^{q+c,\bullet}_{Z_x}))$ and the direct limit runs through non-empty opens $V\subset Z_x$.

The inequalities in (3.8) guarantee that $p\leq j+s-q<c$. Since the complex $\mathcal{E}^{q+c,\bullet}_{Z_x}$ is concentrated in non-negative degrees, the canonical isomorphism (3.9) in turn implies the desired vanishing result (3.8). The proof of Lemma 3.2 is finally complete. □

Corollary 3.3. *For all integers i,s, there is a canonical isomorphism*

$$H^i(X,\tau_{\geq s}K^\bullet)\xrightarrow{\simeq}H^i_{i-s,nr}(X,\tau_{\geq s}K^\bullet).$$

Proof. We apply repeatedly Lemma 3.2 until we reach an isomorphism

$$H^i(F_jX,\tau_{\geq s}K^\bullet)\xrightarrow{\simeq}H^i_{i-s,nr}(X,\tau_{\geq s}K^\bullet)$$

with $j>\max\{\dim X,i-s\}$. The result now follows as $F_jX=X$ for $j\geq\dim X$. □

Lemma 3.4. *We have $H^i(F_jX,\tau_{\leq s}K^\bullet)=0$ for all $j<i-s$.*

Proof. By exactness of the direct limit functor, we get as before a hypercohomology spectral sequence

$$E_2^{p,q}:=H^p(F_jX,\mathcal{H}^q(\tau_{\leq s}K^\bullet))\implies H^{p+q}(F_jX,\tau_{\leq s}K^\bullet).$$

Even though $\tau_{\leq s}K^\bullet$ may be unbounded to the left, the above spectral sequence converges as the Zariski topology on X has finite cohomological dimension, see [G57, Théorème 3.6.5]. It thus suffices to show that

$$H^p(F_jX,\mathcal{H}^q(K^\bullet))=0 \text{ for } p+q=i,\ q\leq s \text{ and } j<i-s.$$

The two inequalities imply that $q<i-j$ and since $p=i-q$ we find $p>j$. The vanishing in question follows from the elementary observation that the complex $\Gamma(F_jX,\mathcal{E}^{q,\bullet}_X)$ is concentrated in degrees $0\leq p\leq j$. Indeed, recall that the sheaf $\mathcal{E}^{q,p}_X$ is a direct sum of sheaves whose supports are closed subvarieties of X of codimension p. Since forming the direct limit commutes with the direct sum, we obtain that $\Gamma(F_jX,\mathcal{E}^{q,p}_X)=0$ for $p>j$, as claimed. □

Corollary 3.5. *The natural map*

$$H^i(F_jX,K^\bullet)\longrightarrow H^i(F_jX,\tau_{\geq s}K^\bullet)$$

is an isomorphism if $j<i-s+1$ and surjective if $j=i-s+1$.

Proof. Consider the canonical exact triangle

$$\tau_{\leq s-1}K^\bullet \longrightarrow K^\bullet \longrightarrow \tau_{\geq s}K^\bullet \xrightarrow{+1}$$

of complexes of Zariski sheaves. This gives in turn an exact sequence

$$H^i(F_jX, \tau_{\leq s-1}K^\bullet) \longrightarrow H^i(F_jX, K^\bullet) \longrightarrow H^i(F_jX, \tau_{\geq s}K^\bullet) \longrightarrow H^{i+1}(F_jX, \tau_{\leq s-1}K^\bullet),$$

where by Lemma 3.4 the first and the last group vanish when $j < i - s + 1$ and $j < i + 1 - s + 1$, respectively. The claim thus follows. ☐

We are now in the position to conclude the proof of Theorem 3.1. We have a commutative diagram

$$\begin{array}{ccc} H^i(F_{i-s+1}X, K^\bullet) & \twoheadrightarrow & H^i(F_{i-s+1}X, \tau_{\geq s}K^\bullet) \\ \downarrow & & \downarrow \\ H^i(F_{i-s}X, K^\bullet) & \xrightarrow{\simeq} & H^i(F_{i-s}X, \tau_{\geq s}K^\bullet), \end{array}$$

where all maps are the canonical ones and where by Corollary 3.5 the upper horizontal map is surjective and the lower one is an isomorphism. This yields an isomorphism

$$H^i_{i-s,nr}(X, K^\bullet) \simeq H^i_{i-s,nr}(X, \tau_{\geq s}K^\bullet).$$

The canonical isomorphism (3.2) is then given by the composition

$$H^i(X, \tau_{\geq s}K^\bullet) \xrightarrow{\simeq} H^i_{i-s,nr}(X, \tau_{\geq s}K^\bullet) \simeq H^i_{i-s,nr}(X, K^\bullet),$$

where the first map is an isomorphism by Corollary 3.3. This completes the proof of Theorem 3.1. ☐

Proof of Theorem 1.7. We replace ν by the Zariski topology and $K^\bullet$ by the total pushforward $R\pi_*K^\bullet$. Then Theorem 3.1 applies and gives the result. ☐

4. Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. Let k, ν and $K^\bullet \in D(X_\nu, \mathbb{Z})$ be as in one of the examples in (1)–(8). For a closed subset $Z \subset X$, we get cohomology theories with support by

$$H^i_Z(X) := \mathrm{R}^i\,\Gamma_Z(X_\nu, K^\bullet),$$

where Γ_Z denotes the global section functor with support. If $j : V \hookrightarrow X$ is an open immersion, then we write

$$H^i_{Z\cap V}(V) := \mathrm{R}^i\,\Gamma_{Z\cap V}(V_\nu, j^*K^\bullet).$$

If $U \subset X$ denotes the complement of $Z \subset X$, then there is a natural exact triangle

$$\mathrm{R}\,\Gamma_Z(X_\nu, -) \longrightarrow \mathrm{R}\,\Gamma(X_\nu, -) \longrightarrow \mathrm{R}\,\Gamma(U_\nu, -) \xrightarrow{+1}$$

defined on $D(X_\nu, \mathbb{Z})$. In particular, we get residue maps $\partial : H^i(U) \to H^{i+1}_Z(X)$. This still works if we replace X by some open subset $V \subset X$ and $K^\bullet$ by its restriction to V. In particular, we get residue maps $\partial : H^i(U \cap V) \to H^{i+1}_{Z\cap V}(V)$.

For a codimension j point $x \in X^{(j)}$ with closure $Z := \overline{\{x\}} \subset X$, we may now define

$$H^i_x(X) := \varinjlim_{x\in V\subset X} H^i_{Z\cap V}(V),$$

where $V \subset X$ runs through all Zariski open subsets of X that contain x. If $V \subset V' \subset X$ are open subsets with $Z' = V' \setminus V$, then the aforementioned residue maps yield a map

$$H^i_{Z\cap V}(V) \longrightarrow H^i(V) \xrightarrow{\partial} H^{i+1}_{Z'}(V').$$

Taking direct limits, we get maps

$$\partial : \bigoplus_{x\in X^{(j)}} H^i_x(X) \longrightarrow \bigoplus_{x\in X^{(j+1)}} H^{i+1}_x(X).$$

For a codimension j point $x \in X^{(j)}$, we define

$$A^{i+j}_x := H^{i+j}_x(X)$$

and we let $\iota_{x*}\underline{A}^{i+j}_x$ be the sheaf on X_{Zar} that is constant equal to A^{i+j}_x on the closure of x and zero outside of that closed subset. The above residue maps thus yield a complex
(4.1)

$$\mathcal{E}^{i,\bullet}_X : \ 0 \longrightarrow \bigoplus_{x\in X^{(0)}} \iota_{x*}\underline{A}^i_x \longrightarrow \bigoplus_{x\in X^{(1)}} \iota_{x*}\underline{A}^{i+1}_x \longrightarrow \bigoplus_{x\in X^{(2)}} \iota_{x*}\underline{A}^{i+2}_x \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(r)}} \iota_{x*}\underline{A}^{i+r}_x \longrightarrow \cdots ,$$

as in (1.4). There is a natural map of complexes $\mathrm{R}^i\,\pi_*K^\bullet \to \mathcal{E}^{i,\bullet}_X$, where we view $\mathrm{R}^i\,\pi_*K^\bullet$ to be concentrated in degree zero.

We need to show that $\mathrm{R}^i\,\pi_*K^\bullet \to \mathcal{E}^{i,\bullet}_X$ is a resolution. It is not hard to see that this follows if one can show a certain effacement theorem, see [Qui73, BO74] and especially [CTHK97, Proposition 2.1.2 and Theorem 2.2.7], which asserts that for any open subset $U \subset X$, any finite set $S \subset U$, and any closed subset $Z \subset U$ of pure codimension j, there is a closed subset $W \subset U$ with $Z \subset W$ and of pure codimension $j-1$, and a closed subset $Z' \subset W$ with $S \cap Z' = \emptyset$, such that the natural composition

$$H^i_Z(U) \longrightarrow H^i_W(U) \longrightarrow H^i_{W\setminus Z'}(U \setminus Z')$$

is zero. Equivalently, the image of $H^i_Z(U) \to H^i_W(U)$ is contained in the image of $H^i_{Z'}(U) \to H^i_W(U)$, where Z' is disjoint from S and hence in "good" position with respect to S. In particular, the effacement theorem may be seen as a special case of a moving lemma for cohomology with support, see [Sch22].

It remains to show that the Gersten conjecture, respectively an "effacement theorem", holds for the cohomology theories considered in (1)–(8). This follows in the case of item (1) from [CTHK97, 7.4(1)], in case of items (2) and (3) from [CTHK97, 7.3(2)–(3)], in the cases of logarithmic de Rham Witt and de Rham Witt cohomology as in items (4) and (5), it follows from [GS88] and [CTHK97, 7.4(3)], and in the case of ℓ-adic pro-étale cohomology it follows from [Sch22, Theorem 1.1 and Proposition 3.2(3)].

Next we consider the case of Bloch's cycle complex tensored with an abelian group A as in item (7) and assume that the field k is perfect. We let

$$\mathbb{Z}^{SF}(n) := \underline{C}_*(z_{equi}(\mathbb{A}^n, 0))[-2n]$$

be the motivic complex of weight n as defined by Friedlander and Suslin in [FS02, Section 8]. Here, $z_{equi}(X, 0)$ denotes the étale sheaf of equi-dimensional cycles on X of relative dimension 0 on the category Sm/k of smooth k-schemes as given in [VSF00, Chapter 4, §2, pp. 141], whereas the complex $\underline{C}_*(F)$

assigned to a presheaf F of abelian groups on Sm/k is the singular simplicial complex of F, see [VSF00, Chapter 4, §4, pp. 150]. In the notation of item (7), we have a canonical isomorphism

$$\mathbb{Z}^{SF}(n)|_{X_{\mathrm{Zar}}} \otimes A \simeq A_X(n)_{\mathrm{Zar}}$$

in the derived category of complexes of sheaves on the small Zariski site X_{Zar}, see [FS02, Proposition 12.1]. In particular, we obtain

$$H^i_M(X, A(n)) \simeq H^i(X_{\mathrm{Zar}}, \mathbb{Z}^{SF}(n) \otimes A).$$

Note that the above tensor product coincides with the derived one, since the sheaves $\underline{C}_*(z_{equi}(\mathbb{A}^n, 0))$ are flat. It can be readily seen that $\mathbb{Z}^{SF}(n) \otimes A$ is the Zariski sheafification of the complex

$$\underline{C}_*(z_{equi}(\mathbb{A}^n, 0) \otimes^{PrSh} A)[-2n],$$

where $\otimes^{PrSh}$ denotes the tensor product in the category of presheaves on Sm/k. The latter can be viewed as a complex of pretheories ([VSF00, Definition 3.1]) with homotopy invariant cohomology presheaves, see [VSF00, Chapter 4, §5, Proposition 5.7] and [VSF00, Chapter 3, §3, Proposition 3.6]. It follows from [VSF00, Chapter 3, §4, Proposition 4.26] that the Zariski cohomology sheaves of $\mathbb{Z}^{SF}(n) \otimes A$ are homotopy invariant pretheories over k and thus their restriction to X_{Zar} admits a Gersten resolution by [VSF00, Chapter3, §4, Theorem 4.37], as required.

Lastly, the case of étale motivic cohomology as in item (8) (i.e. with coefficients in an abelian group A in which the characteristic exponent p is invertible) is essentially contained in [CD16]. For the reader's convenience we include some details of the argument. Let $\mathrm{DM}_{\mathrm{h}}(X, A)$ denote the category of h-motives as defined in [CD16, Definition 5.1.3]. The authors in [CD16] define étale motivic cohomology by

$$H^i(X, n) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,A)}(A_X, A_X(n)[i]),$$

where A_X is the identity object for $\otimes$ in $\mathrm{DM}_{\mathrm{h}}(X, A)$ and $A_X(1)$ is the Tate object. The above definition agrees with Lichtenbaum cohomology if the characteristic exponent p of k is invertible in A and X is a smooth and equi-dimensional algebraic k-scheme, see [CD16, Theorem 7.1.2]. We show that the cohomology theory with supports

$$H^i_Z(X, n) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(Z,A)}(A_Z, \iota^! A_X(n)[i]) \tag{4.2}$$

satisfies the effacement theorem, where $\iota : Z \hookrightarrow X$ is a closed embedding of algebraic k-schemes. By [CD16, Theorem 5.6.2], the triangulated premotivic category $\mathrm{DM}_{\mathrm{h}}(-, A)$ satisfies the formalism of the Grothendieck 6 functors for Noetherian schemes of finite dimension ([CD16, Definition A.1.10]) along with the absolute purity property ([CD16, Definition A.2.9]). This ensures as in the case of étale cohomology with finite coefficients [BO74, Example (2.1)] that the axioms from [BO74, Definition (1.1) & (1.2)] are satisfied and so the effacement theorem holds, as we want.

Altogether, this completes the proof of the theorem. □

5. Applications

In this section we prove Corollaries 1.3, 1.4, 1.5 and 1.6.

5.1. Comparison to the cycle class map on higher Chow groups with finite coefficients.

Proof of Corollary 1.3. Let k be a perfect field and let $\pi : X_{\text{ét}} \to X_{\text{Zar}}$ be the natural map of sites from the étale site to the Zariski site of X and let m be an arbitrary integer. Let $(\mathbb{Z}/m)_X(n)_{\text{Zar}} := z^n(-_{\text{Zar}}, \bullet)[-2n]\otimes^{\mathbb{L}}\mathbb{Z}/m$ denote Bloch's cycle complex from [Blo86] tensored with $\mathbb{Z}/m$ and let $(\mathbb{Z}/m)_X(n)_{\text{ét}}$ be its étale sheafification.

By [GL00, GL01], we have canonical isomorphisms in $D(X_{\text{ét}}, \mathbb{Z})$

$$(\mathbb{Z}/m)_X(n)_{\text{ét}} \simeq \begin{cases} \mu_m^{\otimes n} & \text{if } m \text{ is coprime to char}(k); \\ W_r\Omega^n_{X,\log}[-n] & \text{if } m = p^r \text{ and } p = \text{char}(k) > 0. \end{cases}$$

The higher direct images via π of the sheaves/complexes on the right admit Gersten resolutions by [BO74, GS88]. It follows from the Chinese remainder theorem that the same holds for $\mathrm{R}^i\,\pi_*(\mathbb{Z}/m)_X(n)_{\text{ét}}$ and all i, m. Hence the assumptions of Theorem 1.7 are satisfied for the complex $(\mathbb{Z}/m)_X(n)_{\text{ét}}$ and we get a canonical isomorphism

$$H^i(X_{\text{Zar}}, \tau_{\geq s}\,\mathrm{R}\,\pi_*(\mathbb{Z}/m)_X(n)_{\text{ét}}) \simeq H^i_{i-s,nr}(X_{\text{ét}}, (\mathbb{Z}/m)_X(n)_{\text{ét}}). \tag{5.1}$$

The cohomology sheaves of the complex $(\mathbb{Z}/m)_X(n)_{\text{Zar}}$ vanish in all degrees greater than n. Indeed, from the Bockstein long exact sequence

$$\cdots \longrightarrow \mathcal{H}^i(\mathbb{Z}(n)_{\text{Zar}}) \xrightarrow{\times m} \mathcal{H}^i(\mathbb{Z}(n)_{\text{Zar}}) \longrightarrow \mathcal{H}^i(\mathbb{Z}/m(n)_{\text{Zar}}) \longrightarrow \mathcal{H}^{i+1}(\mathbb{Z}(n)_{\text{Zar}}) \xrightarrow{\times m} \cdots,$$

we see that it is enough to show that $\mathcal{H}^i(\mathbb{Z}(n)_{\text{Zar}}) = 0$ for all $i > n$. The latter is implied by the Gersten conjecture [Blo86, Theorem 10.1] and the fact that $\text{CH}^n(\operatorname{Spec} k(x), 2n-i) = 0$ for all $i > n$ and all points $x \in X$. Thus the canonical map $(\mathbb{Z}/m)_X(n)_{\text{Zar}} \to \mathrm{R}\,\pi_*(\mathbb{Z}/m)_X(n)_{\text{ét}}$ factors through the truncation $\tau_{\leq n}\,\mathrm{R}\,\pi_*\mu_m^{\otimes n}$. By the Beilinson-Lichtenbaum conjecture proven by Voevodsky [Voe03, Voe11], we get in turn a natural quasi-isomorphism

$$(\mathbb{Z}/m)_X(n)_{\text{Zar}} \simeq \tau_{\leq n}\,\mathrm{R}\,\pi_*(\mathbb{Z}/m)_X(n)_{\text{ét}}. \tag{5.2}$$

Indeed, by the Chinese remainder theorem, it suffices to prove the above for m a power of a prime number. In the case m is coprime to the characteristic, the claim follows from [GL01, Corollary 1.2] and the Bloch-Kato conjecture [Voe11] (see also [Voe03, Theorem 6.6]) whereas if m is a power of the characteristic from [GL00, Theorem 8.5].

We now consider the canonical exact triangle

$$\tau_{\leq n}\,\mathrm{R}\,\pi_*(\mathbb{Z}/m)_X(n)_{\text{ét}} \longrightarrow \mathrm{R}\,\pi_*(\mathbb{Z}/m)_X(n)_{\text{ét}} \longrightarrow \tau_{\geq n+1}\,\mathrm{R}\,\pi_*(\mathbb{Z}/m)_X(n)_{\text{ét}} \xrightarrow{+1}$$

in $D(X_{\text{Zar}}, \mathbb{Z})$. The result follows then if we apply $\mathrm{R}\,\Gamma(X_{\text{Zar}}, -)$ to this triangle, consider the cohomology sequence of the resulting triangle and use (5.1) and (5.2). This concludes the proof of the corollary. □

5.2. Refined unramified cohomology in the log de Rham Witt case.

Proof of Corollary 1.4. By Theorem 1.2, we have a canonical isomorphism

$$H^i_{j,nr}(X_{\text{ét}}, (\mathbb{Z}/p^r)_X(n)) \simeq H^i(X_{\text{Zar}}, \tau_{\geq i-j}\,\mathrm{R}\,\pi_*(\mathbb{Z}/p^r)_X(n)).$$

From [GS88, Corollaire 1.5] we find that $\mathrm{R}^q\,\pi_*(\mathbb{Z}/p^r)_X(n) = 0$ for all $q \neq n, n+1$. This implies in turn that for $n \geq i-j$ the canonical map $\mathrm{R}\,\pi_*(\mathbb{Z}/p^r)_X(n) \to \tau_{\geq i-j}\,\mathrm{R}\,\pi_*(\mathbb{Z}/p^r)_X(n)$ is a quasi-isomorphism as well as for $n < i-j-1$ the truncated complex $\tau_{\geq i-j}\,\mathrm{R}\,\pi_*(\mathbb{Z}/p^r)_X(n)$ is quasi-isomorphic to zero. Finally if $n = i-j-1$, then the (p,q) terms with $q \neq i-j$ of the spectral sequence

$$E_2^{p,q} := H^p(X_{\mathrm{Zar}}, \mathcal{H}^q(\tau_{\geq i-j}\,\mathrm{R}\,\pi_*(\mathbb{Z}/p^r)_X(n))) \implies H^{p+q}(X_{\mathrm{Zar}}, \tau_{\geq i-j}\,\mathrm{R}\,\pi_*(\mathbb{Z}/p^r)_X(n))$$

all vanish and thus $H^i_{j,nr}(X, (\mathbb{Z}/p^r)_X(n)) \simeq H^j(X_{\mathrm{Zar}}, \mathrm{R}^{i-j}\,\pi_*(\mathbb{Z}/p^r)_X(n))$ as we want. This finishes the proof. $\square$

Remark 5.1. *The Leray spectral sequence associated to $\pi : X_{\acute{e}t} \to X_{Zar}$,*

$$E_2^{p,q} := H^p(X_{Zar}, \mathrm{R}^q\,\pi_*(\mathbb{Z}/p^r)_X(n)) \implies H^{p+q}(X, (\mathbb{Z}/p^r)_X(n)),$$

together with the fact that $\mathrm{R}^q\,\pi_*(\mathbb{Z}/p^r)_X(n) = 0$ for $q \neq n, n+1$ (see [GS88, Corollaire 1.5]*) yield an exact sequence*

$$\begin{aligned} &0 \longrightarrow H^1(X_{Zar}, \mathrm{R}^n\,\pi_*(\mathbb{Z}/p^r)_X(n)) \longrightarrow H^{n+1}(X, (\mathbb{Z}/p^r)_X(n)) \\ &\longrightarrow H^0(X_{Zar}, \mathrm{R}^{n+1}\,\pi_*(\mathbb{Z}/p^r)_X(n)) \longrightarrow \cdots \longrightarrow H^n(X_{Zar}, \mathrm{R}^n\,\pi_*(\mathbb{Z}/p^r)_X(n)) \\ &\longrightarrow H^{2n}(X, (\mathbb{Z}/p^r)_X(n)) \longrightarrow H^{n-1}(X_{Zar}, \mathrm{R}^{n+1}\,\pi_*(\mathbb{Z}/p^r)_X(n)) \longrightarrow 0. \end{aligned} \tag{5.3}$$

Since by (5.2) *we also have $H^i_M(X, \mathbb{Z}/p^r(n)) \simeq H^{i-n}(X_{Zar}, \mathrm{R}^n\,\pi_*(\mathbb{Z}/p^r)_X(n))$, we find from Corollary 1.4 that the sequence* (5.3) *identifies with the one of Corollary 1.3.*

5.3. **Pullbacks for refined unramified cohomology.**

Proof of Corollary 1.5. Let $f : X \to Y$ be a morphism between equi-dimensional smooth algebraic k-schemes and let k, ν and $K_Y^\bullet \in D(Y_\nu, \mathbb{Z})$ be as in one of the examples (1)–(8) above. Let $K_X^\bullet \in D(X_\nu, \mathbb{Z})$ be the analogous complex on X_ν. We aim to construct functorial pullback maps

$$f^* : H^i_{j,nr}(Y_\nu, K_Y^\bullet) \longrightarrow H^i_{j,nr}(X_\nu, K_X^\bullet).$$

To this end, let us first construct a natural map $f^* : H^i_{j,nr}(Y_\nu, K_Y^\bullet) \to H^i_{j,nr}(X_\nu, f^*K_Y^\bullet)$. By Theorem 1.2, this is equivalent to establishing a natural map

$$f^* : H^i(Y_{\mathrm{Zar}}, \tau_{\geq i-j}\,\mathrm{R}\,\pi_{Y*}K_Y^\bullet) \longrightarrow H^i(X_{\mathrm{Zar}}, \tau_{\geq i-j}\,\mathrm{R}\,\pi_{X*}f^*K_Y^\bullet), \tag{5.4}$$

where $\pi_Y : Y_\nu \to Y_{\mathrm{Zar}}$ (resp. $\pi_X : X_\nu \to X_{\mathrm{Zar}}$) denotes the natural morphism of sites. The unit $\alpha_f : \mathrm{id} \to \mathrm{R}\,f_*f^*$ of the adjunction (f^*, f_*) gives a map

$$\alpha_f : \tau_{\geq i-j}\,\mathrm{R}\,\pi_{Y*}K_Y^\bullet \longrightarrow \mathrm{R}\,f_*f^*\tau_{\geq i-j}\,\mathrm{R}\,\pi_{Y*}K_Y^\bullet.$$

Since $f^* : Shv(Y_{\mathrm{Zar}}, \mathbb{Z}) \to Shv(X_{\mathrm{Zar}}, \mathbb{Z})$ is an exact functor, it induces a functor on the derived level that we denote by the same symbol f^* and which commutes with truncation. In particular, the target of α_f is $\mathrm{R}\,f_*\tau_{\geq i-j}f^*\,\mathrm{R}\,\pi_{Y*}K_Y^\bullet$. We exhibit a natural map

$$\beta : f^*\,\mathrm{R}\,\pi_{Y*}K_Y^\bullet \longrightarrow \mathrm{R}\,\pi_{X*}f^*K_Y^\bullet.$$

By adjunction the latter is equivalent to constructing a map

$$\beta^\# : \mathrm{R}\,\pi_{Y*}K_Y^\bullet \longrightarrow \mathrm{R}\,f_*\,\mathrm{R}\,\pi_{X*}f^*K_Y^\bullet.$$

Note that $\mathrm{R}\, f_* \mathrm{R}\, \pi_{X*} = \mathrm{R}(f \circ \pi_X)_* = \mathrm{R}(\pi_Y \circ f)_* = \mathrm{R}\, \pi_{Y*} \mathrm{R}\, f_*$ and hence the target of $\beta^{\#}$ is really $\mathrm{R}\, \pi_{Y*} \mathrm{R}\, f_* f^* K_Y^\bullet$. To obtain the map $\beta^{\#}$, we simply apply now the derived functor $\mathrm{R}\, \pi_{Y*}$ to the natural map $K_Y^\bullet \to \mathrm{R}\, f_* f^* K_Y^\bullet$ obtained by evaluating the unit of the adjunction (f^*, f_*) at $K_Y^\bullet$.

Consider the composite

$$\tau_{\geq i-j} \mathrm{R}\, \pi_{Y*} K_Y^\bullet \xrightarrow{\alpha_f} \mathrm{R}\, f_* \tau_{\geq i-j} f^* \mathrm{R}\, \pi_{Y*} K_Y^\bullet \xrightarrow{\mathrm{R}\, f_* \tau_{\geq i-j} \beta} \mathrm{R}\, f_* \tau_{\geq i-j} \mathrm{R}\, \pi_{X*} f^* K_Y^\bullet.$$

We then obtain the map (5.4) as desired by applying $\mathrm{R}^i\, \Gamma(Y_{\mathrm{Zar}}, -)$ to the above composition.

Next, we need to show that there exists a natural map $f^* K_Y^\bullet \to K_X^\bullet$ for each one of the examples (1)–(8). The examples (1) and (6) are of the same nature i.e. the complexes in question are given by $K_X^\bullet := p_X^* K^\bullet$ and $K_Y^\bullet := p_Y^* K^\bullet$, where $p_X : X \to \operatorname{Spec} k$ (resp. $p_Y : Y \to \operatorname{Spec} k$) is the structure morphism of X (resp. Y) and where $K^\bullet \in D(\operatorname{Spec}(k)_\nu)$. Since $f^* p_Y^* \simeq p_X^*$, the claim follows. For (2), note that locally any complex of locally constant sheaves of abelian groups in $D(X_{\mathrm{an}})$ is a pullback of a complex from $D(\operatorname{Spec}(\mathbb{C})_{\mathrm{an}}) = D(\mathrm{Ab})$. Thus the same reasoning as before implies the result. For (3), see [Stacks24, Tag 0FKL] and for (4) and (5), see [G85, pp. 10, (1.2.2)] and [Il79, pp. 548, (1.12.3)], respectively.

It remains to treat the case of items (7) and (8). As in the proof of Theorem 1.2, recall that by [FS02, Proposition 12.1], there is a canonical quasi-isomorphism of complexes of Zariski sheaves on the small site of étale Y-schemes:

$$\mathbb{Z}^{SF}(n)|_{Y_{\text{ét}}} \otimes A \simeq A_Y(n),$$

where $\mathbb{Z}^{SF}(n) = \underline{C}_*(z_{equi}(\mathbb{A}^n, 0))[-2n]$ is the motivic complex of Friedlander and Suslin given in [FS02, Section 8]. As before, the presheaf $z_{equi}(X, 0)$ on Sm/k is the étale sheaf of equi-dimensional cycles on X of relative dimension 0 as defined in [VSF00, Chapter 4, §2, pp. 141], whereas the complex $\underline{C}_*(F)$ associated to a presheaf F of abelian groups on Sm/k is the singular simplicial complex of F, see [VSF00, Chapter 4, §4, pp. 150]. In particular, we can work with the motivic complex $\mathbb{Z}^{SF}(n)$ instead of Bloch's cycle complex. It is readily seen that by contravariant functoriality of $z_{equi}(X, 0)$, we obtain maps

$$\underline{C}_*(z_{equi}(\mathbb{A}^n, 0))|_{Y_\nu} \longrightarrow f_* \underline{C}_*(z_{equi}(\mathbb{A}^n, 0))|_{X_\nu}$$

and especially a natural map $\mathbb{Z}^{SF}(n)|_{Y_\nu} \to f_* \mathbb{Z}^{SF}(n)|_{X_\nu}$. The adjoint of the latter gives naturally what we want.

Lastly, note that we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{R}\, \pi_{Y*} K_Y^\bullet & \xrightarrow{\alpha_f} & \mathrm{R}\, f_* f^* \mathrm{R}\, \pi_{Y*} K_Y^\bullet & \xrightarrow{\mathrm{R}\, f_* \beta} & \mathrm{R}\, f_* \mathrm{R}\, \pi_{X*} f^* K_Y^\bullet & \longrightarrow & \mathrm{R}\, f_* \mathrm{R}\, \pi_{X*} K_X^\bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\geq i-j} \mathrm{R}\, \pi_{Y*} K_Y^\bullet & \xrightarrow{\alpha_f} & \mathrm{R}\, f_* \tau_{\geq i-j} f^* \mathrm{R}\, \pi_{Y*} K_Y^\bullet & \xrightarrow{\mathrm{R}\, f_* \tau_{\geq i-j} \beta} & \mathrm{R}\, f_* \tau_{\geq i-j} \mathrm{R}\, \pi_{X*} f^* K_Y^\bullet & \longrightarrow & \mathrm{R}\, f_* \tau_{\geq i-j} \mathrm{R}\, \pi_{X*} K_X^\bullet,
\end{array}$$

where the vertical maps are the canonical ones. Applying $\mathrm{R}\, \Gamma(Y_{\mathrm{Zar}}, -)$ to the above diagram and taking cohomology, we obtain a commutative diagram:

$$\begin{array}{ccc}
H^i(Y_\nu, K_Y^\bullet) & \xrightarrow{f^*} & H^i(X_\nu, K_X^\bullet) \\
\downarrow & & \downarrow \\
H^i(Y_{\mathrm{Zar}}, \tau_{\geq i-j} \mathrm{R}\, \pi_{Y*} K_Y^\bullet) & \xrightarrow{f^*} & H^i(X_{\mathrm{Zar}}, \tau_{\geq i-j} \mathrm{R}\, \pi_{X*} K_X^\bullet).
\end{array}$$

The vertical maps identify by Theorem 1.2 with the natural maps $H^i(Y_\nu, K_Y^\bullet) \to H^i_{j,nr}(Y_\nu, K_Y^\bullet)$ and $H^i(X_\nu, K_X^\bullet) \to H^i_{j,nr}(X_\nu, K_X^\bullet)$, respectively. When $j \geq \max\{\dim X, \dim Y\}$, they become isomorphisms simply because $F_j X = X$ (resp. $F_j Y = Y$). This concludes the proof of Corollary 1.5, as we want. □

5.4. **Action of correspondences.** In this section we prove Corollary 1.6. Let X be a smooth and equi-dimensional algebraic scheme over a perfect field k of characteristic $p > 0$. Recall first that by works of Berthelot (see [Be74, Chapter VI, §3]) and Gros (see [G85, Définition 4.1.7]), there is a well-defined cycle class map

$$\mathrm{cl}_X^c : \mathrm{CH}^c(X) \longrightarrow H^{2c}(X, W\Omega_X^\bullet) \tag{5.5}$$

that satisfies various natural compatibility properties.

We start with some preliminary lemmas.

Lemma 5.2. *Let X be a smooth equi-dimensional algebraic scheme over a perfect field k of characteristic $p > 0$. Then for any pair of smooth and equi-dimensional algebraic k-schemes X and Y, there is a natural bi-additive pairing*

$$\times : \mathrm{CH}^c(X) \times H^i_{j,nr}(Y, W\Omega_Y^\bullet) \longrightarrow H^{i+2c}_{j+c,nr}(X \times Y, W\Omega^\bullet_{X\times Y}), \tag{5.6}$$

which coincides with the pairing

$$\mathrm{CH}^c(X) \times H^i(Y, W\Omega_Y^\bullet) \longrightarrow H^{i+2c}(X \times Y, W\Omega^\bullet_{X\times Y}),\ \ [\Gamma] \otimes \alpha \longmapsto p^* \mathrm{cl}_X^c([\Gamma]) \cup q^* \alpha,$$

if $j \geq \dim Y + \dim X - c$, where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the canonical projections.

Proof. Let $\mathcal{Z}^c(X)$ denote the free abelian group generated by integral closed subschemes of X of codimension c. As a first step, we construct a bi-additive pairing

$$\mathcal{Z}^c(X) \times H^i(F_j Y, W\Omega_Y^\bullet) \longrightarrow H^{i+2c}(F_{j+c}(X \times Y), W\Omega^\bullet_{X\times Y}). \tag{5.7}$$

Pick a cycle $\Gamma \in \mathcal{Z}^c(X)$ and denote its support by $|\Gamma| := \mathrm{Supp}(\Gamma) \subset X$. Let $\alpha \in H^i(F_j Y, W\Omega_Y^\bullet)$ be any class and choose a lift $\tilde{\alpha} \in H^i(U, W\Omega_Y^\bullet)$, where $U \subset Y$ is open with complement $R := Y \setminus U$ of codimension $\geq j+1$. Consider the cycle class $\mathrm{cl}_{|\Gamma|}(\Gamma) \in H^{2c}_{|\Gamma|}(X, W\Omega_X^\bullet)$ and note that it behaves well with respect to pull-backs. Indeed, by [G85, Définition 4.1.7], it suffices to prove the compatibility for the cohomology class $\mathrm{cl}_{|\Gamma|}(\Gamma) \in H^c_{|\Gamma|}(X, W\Omega^c_{X,log})$. The natural isomorphism [G85, (4.1.6)], allows us to remove the singular locus of $|\Gamma|$. Thus the compatibility property in question follows from the commutative diagram [G85, (3.5.20)], once we use the identification [G85, (3.5.19)]. In particular, if $p : X \times Y \to X$ is the natural projection, then $p^* \mathrm{cl}_{|\Gamma|}(\Gamma) = \mathrm{cl}_{|\Gamma|\times Y}(p^*\Gamma) \in H^{2c}_{|\Gamma|\times Y}(X \times Y, W\Omega^\bullet_{X\times Y})$.

We consider the following composite

(5.8)

$$H^i(U, W\Omega_Y^\bullet) \overset{\mathrm{cl}_{|\Gamma|\times U}(p^*\Gamma)\cup q^*}{\longrightarrow} H^{i+2c}_{|\Gamma|\times U}((X\times Y)\setminus(|\Gamma|\times R), W\Omega^\bullet_{X\times Y}) \overset{\iota_*}{\longrightarrow} H^{i+2c}((X\times Y)\setminus(|\Gamma|\times R), W\Omega^\bullet_{X\times Y}),$$

where for the middle term we implicitly use the isomorphism

$$H^{i+2c}_{|\Gamma|\times U}(X \times U, W\Omega^\bullet_{X\times Y}) \simeq H^{i+2c}_{|\Gamma|\times U}((X \times Y) \setminus (|\Gamma| \times R), W\Omega^\bullet_{X\times Y})$$

that follows from excision. The closed subset $|\Gamma| \times R \subset X \times Y$ has codimension $\geq j + c + 1$ in $X \times Y$ and thus we can define $\Gamma \times \alpha \in H^{i+2c}(F_{j+c}(X \times Y), W\Omega^\bullet_{X\times Y})$ as the class represented by the image of

$\tilde{\alpha}$ via (5.8). Note that the class $\Gamma \times \alpha$ is clearly independent of our choice of a lift $\tilde{\alpha}$, as we can always shrink $U \subset X$ to an open subset $F_jY \subset U' \subset U$ over which a given lift of α agrees with $\tilde{\alpha}|_{U'}$.

By construction, the diagram

$$\begin{array}{ccc} \mathcal{Z}^c(X) \times H^i(F_{j+1}Y, W\Omega_Y^\bullet) & \longrightarrow & H^i(F_{j+c+1}(X \times Y), W\Omega_{X\times Y}^\bullet) \\ \downarrow & & \downarrow \\ \mathcal{Z}^c(X) \times H^i(F_jY, W\Omega_Y^\bullet) & \longrightarrow & H^i(F_{j+c}(X \times Y), W\Omega_{X\times Y}^\bullet) \end{array}$$

commutes, thus inducing a bi-additive pairing

$$\times : \mathcal{Z}^c(X) \times H^i_{j,nr}(Y, W\Omega_Y^\bullet) \longrightarrow H^{i+2c}_{j+c,nr}(X \times Y, W\Omega_{X\times Y}^\bullet).$$

To get (5.6), it remains to show that the class $\Gamma \times \alpha$ vanishes if Γ is rationally equivalent to zero. Indeed, in this case there exists an equi-dimensional closed subset $Z \subset X$ of codimension $c-1$, such that $|\Gamma| \subset Z$ and $\mathrm{cl}_Z(\Gamma) = 0 \in H^{2c}_Z(X, W\Omega_X^\bullet)$. Let $\tilde{\alpha} \in H^i(V, W\Omega_Y^\bullet)$ be a lift of $\alpha \in H^i_{j,nr}(Y, W\Omega_Y^\bullet)$ over an open subset $V \subset Y$ whose complement $D := Y \setminus V$ has codimension $\geq j+2$ and consider the composite

(5.9)
$$H^i(V, W\Omega_Y^\bullet) \xrightarrow{\mathrm{cl}_{Z\times V}(p^*\Gamma)\cup q^*} H^{i+2c}_{Z\times V}((X\times Y)\setminus(Z\times D), W\Omega_{X\times Y}^\bullet) \xrightarrow{\iota_*} H^{i+2c}((X\times Y)\setminus(Z\times D), W\Omega_{X\times Y}^\bullet),$$

where as before we make use of the excision isomorphism

$$H^{i+2c}_{Z\times V}((X\times Y)\setminus(Z\times D), W\Omega_{X\times Y}^\bullet) \simeq H^{i+2c}_{Z\times V}(X\times V, W\Omega_{X\times Y}^\bullet)$$

for the middle term.

Note that the closed subset $Z \times D \subset X \times Y$ has codimension $\geq j+c+1$ and it is readily seen that the image of $\tilde{\alpha} \in H^i(V, W\Omega_Y^\bullet)$ via (5.9) gives a representative for the class $\Gamma \times \alpha$. Since the second map in (5.9) is zero ($\mathrm{cl}_Z(\Gamma) = 0$), we find that $\Gamma \times \alpha = 0$, as claimed. □

Lemma 5.3. *Let k be a perfect field of characteristic $p > 0$. Let $f : X \to Y$ be a proper morphism between smooth and equi-dimensional algebraic schemes over k and set $r := \dim Y - \dim X$. Then for any integers $i, j \geq 0$, there is a natural pushforward map $f_* : H^i_{j,nr}(X, W\Omega_X^\bullet) \to H^{i+2r}_{j+r,nr}(Y, W\Omega_Y^\bullet)$ that coincides with $f_* : H^i(X, W\Omega_X^\bullet) \to H^{i+2r}(Y, W\Omega_Y^\bullet)$ for $j \geq \dim X$.*

Proof. It is enough to construct for $j \geq 0$ natural maps $f_* : H^i(F_jX, W\Omega_X^\bullet) \to H^{i+2r}(F_{j+r}Y, W\Omega_Y^\bullet)$ that fit into a commutative diagram

(5.10)
$$\begin{array}{ccc} H^i(F_{j+1}X, W\Omega_X^\bullet) & \xrightarrow{f_*} & H^{i+2r}(F_{j+1+r}Y, W\Omega_Y^\bullet) \\ \downarrow & & \downarrow \\ H^i(F_jX, W\Omega_X^\bullet) & \xrightarrow{f_*} & H^{i+2r}(F_{j+r}Y, W\Omega_Y^\bullet). \end{array}$$

By [G85, Definition 1.2.1], there is a natural morphism

(5.11)
$$f_* : \mathrm{R}\, f_* W\Omega_X^\bullet \longrightarrow W\Omega_Y^\bullet(r)[r]$$

in $D^+(X_{\mathrm{Zar}}, R)$, i.e. in the derived category of Zariski sheaves of graded modules over the Raynaud ring R, see [Il83, (2.1)]. Here $W\Omega_Y^\bullet(r)$ is the graded R-module deduced by the usual shift of degrees i.e. $(W\Omega^\bullet_Y(r))^a = W\Omega^{a+r}_Y$ and the change of the differential d to $(-1)^r d$. There is a derived functor $\mathrm{R}\,\Gamma : D^+(X_{\mathrm{Zar}}, R) \to D^+(R)$ for the global section functor Γ. As we view complexes in $D^+(R)$ (resp. $D^+(X_{\mathrm{Zar}}, R)$) as double complexes whose rows correspond to graded R-modules, we also have a functor $\underline{s} : D^+(R) \to D^+(W)$ (resp. $\underline{s} : D^+(X_{\mathrm{Zar}}, R) \to D^+(X_{\mathrm{Zar}}, W)$) that maps a double complex to its total complex, see [Il83, (2.1)].

Applying $\underline{s} \circ \mathrm{R}\,\Gamma$ to (5.11) gives

$$\underline{s}\,\mathrm{R}\,\Gamma(X, W\Omega^\bullet_X) \longrightarrow \underline{s}\,\mathrm{R}\,\Gamma(Y, W\Omega^\bullet_Y)[2r],$$

where we implicitly utilise the isomorphism of total complexes $\underline{s}\,\mathrm{R}\,\Gamma(Y, W\Omega^\bullet_Y)[2r] \simeq \underline{s}\,\mathrm{R}\,\Gamma(Y, W\Omega^\bullet_Y(r)[r])$ defined degree-wise on the summands by

$$(-1)^{pr}\,\mathrm{id} : \mathrm{R}\,\Gamma(Y, W\Omega^p_Y)^q \longrightarrow \mathrm{R}\,\Gamma(Y, W\Omega^p_Y)^q.$$

Thus taking cohomology in turn yields a natural map $f_* : H^i(X, W\Omega^\bullet_X) \to H^{i+2r}(Y, W\Omega^\bullet_Y)$ for all i.

Next pick a closed subset $Z \subset X$ with $\dim X - \dim Z \geq j+1$ and set $U := X \setminus Z$. Then we have $\dim Y - \dim f(Z) \geq r + j + 1$ and since the map $f : X \setminus f^{-1}(f(Z)) \to X \setminus f(Z)$ is proper, the discussion above yields a homomorphism

$$H^i(U, W\Omega^\bullet_X) \longrightarrow H^i(X \setminus f^{-1}(f(Z)), W\Omega^\bullet_X) \xrightarrow{f_*} H^{i+2r}(Y \setminus f(Z), W\Omega^\bullet_Y) \longrightarrow H^{i+2r}(F_{j+r}Y, W\Omega^\bullet_Y).$$

By taking the limit now over all opens $F_jX \subset U \subset X$, we deduce the desired map

$$f_* : H^i(F_jX, W\Omega^\bullet_X) \longrightarrow H^{i+2r}(F_{j+r}Y, W\Omega^\bullet_Y).$$

The compatibility property (5.10) for f_* is clear by construction. This finishes the proof. □

By combining Lemma 5.2 and Corollary 1.5, we are able to prove Corollary 1.6 stated in the introduction.

Proof of Corollary 1.6. Let k be a perfect field of characteristic $p > 0$ and let X and Y be smooth, proper and equi-dimensional algebraic schemes over k. We aim to construct a bi-additive pairing

$$\mathrm{CH}^c(X \times Y) \times H^i_{j,nr}(X, W\Omega^\bullet_X) \longrightarrow H^{i+2c-2d_X}_{j+c-d_X,nr}(Y, W\Omega^\bullet_Y), \quad ([\Gamma], \alpha) \longmapsto [\Gamma]_*(\alpha), \tag{5.12}$$

where $d_X = \dim X$. We define (5.12) as the composite

$$\mathrm{CH}^c(X \times Y) \times H^i_{j,nr}(X, W\Omega^\bullet_X) \xrightarrow{\Delta^* \circ (5.6) \circ (\mathrm{id} \times p^*)} H^{i+2c}_{j+c,nr}(X \times Y, W\Omega^\bullet_{X\times Y}) \xrightarrow{q_*} H^{i+2c-2d_X}_{j+c-d_X,nr}(Y, W\Omega^\bullet_Y),$$

where $p : X \times Y \to X, q : X \times Y \to Y$ are the natural projections and the morphism $\Delta : X \times Y \to (X \times Y)^2$ denotes the diagonal. Note that in the above composition the pullback maps p^* and Δ^* are well-defined by Corollary 1.5 and that q_* is the pushforward map from Lemma 5.3. Finally an adaptation of the argument in [Sch22, Corollary 6.8 (3)] implies that the construction (1.3) is compatible with respect to the composition of correspondences. □

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